

Smooth Sums over Smooth k -Free Numbers and Statistical Mechanics

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Abstract

We provide an asymptotic estimate for certain sums over k -free integers with small prime factors. These sums depend upon a complex parameter α and involve a smooth cut-off f . They are a variation of several classical number-theoretical sums. One term in the asymptotics is an integral operator whose kernel is the α -convolution of the Dickman-de Bruijn distribution, and the other term is explicitly estimated. The trade-off between the value of α and the regularity of f is discussed. This work generalizes the results of [5, 6], where $k = 2$ and $\alpha = 1$.

Keywords: k -free numbers, smooth-numbers, average order of arithmetic functions, convolutions of the Dickman-De Bruijn distribution, weak convergence of complex measures. **MSC:** 11N37, 11K65, 60B10, 60F05.

1 Introduction

The study of the typical behavior of arithmetic functions has a long history in number theory. Let n denote a positive integer. Let $\omega(n)$ (resp. $\Omega(n)$) denote the number of prime divisors of n , counted without (resp. with) multiplicity. If $d(n)$ denotes the number of divisors of n , then clearly $\Omega(n) \leq \log n / \log 2$, and $2^{\omega(n)} \leq d(n) \leq 2^{\Omega(n)} \leq n$. Notice that $2^{\omega(n)}$ equals the number of square-free divisors of n .

We are interested in k -free numbers, i.e. integers such that $p^k \nmid n$ for every prime p . Notice that for $k = 2$ the set of square-free numbers is characterized by the condition $\Omega(n) = \omega(n)$. This paper is devoted to the study of the asymptotic behavior of certain sums over k -free numbers, with an additional restriction on the size of their prime factors. Let us fix an integer $k \geq 2$, $\alpha \in \mathbb{C}$, and a bounded function $f : \mathbb{R} \rightarrow \mathbb{C}$. Define the sum

$$S_{\Omega,f}(k, \alpha; N) = \sum_{\substack{n \text{ } k\text{-free} \\ p|n \Rightarrow p \leq N}} f\left(\frac{\log n}{\log N}\right) \frac{\alpha^{\Omega(n)}}{n}. \quad (1)$$

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We shall refer to (1) as a *smooth* sum because the regularity of f plays an important role in our analysis. The k -free numbers involved in the sum classically called *smooth* because their prime factors are considerably smaller than the numbers themselves.

We shall always assume that $\alpha \neq 0$, otherwise there is nothing to prove. Let us define the function space

$$\mathcal{S}_\eta(\mathbb{R}) = \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : \exists C > 0 \text{ s.t. } \left| \hat{f}(\lambda) \right| \leq \frac{C}{1 + |\lambda|^\eta} \forall \lambda \in \mathbb{R} \right\},$$

where \hat{f} is the Fourier transform of f . Notice that, for example, $\mathbf{1}_{[0,1]} \in \mathcal{S}_1$ and that the Schwartz space $\mathcal{S} \subset \mathcal{S}_\eta$ for every η . The main result of our paper is the following

Theorem 1.1. *Let $|\alpha| < 2$ and let $f \in \mathcal{S}_\eta$ with $\eta > |\alpha| - \Re \alpha + 1$. Then there exists a non-zero constant $C = C(k, \alpha) \in \mathbb{C}$ such that, for every $R = R(N)$ satisfying*

$$\frac{R}{\log N} \rightarrow 0 \quad \text{and} \quad \frac{(\log N)^{|\alpha| - \Re \alpha}}{R^{\eta-1}} \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

we have

$$S_{\Omega, f}(k, \alpha; N) = C \cdot (\log N)^\alpha \cdot \left(\int_{|\lambda| \leq R} \varphi^{(\alpha)}(\lambda) \hat{f}(\lambda) d\lambda + \varepsilon_N \right), \quad (2)$$

where

$$\varphi^{(\alpha)}(\lambda) = \exp \left\{ \alpha \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv \right\} \quad (3)$$

is the characteristic function (inverse Fourier transform) of the α -convolution of the Dickman-de Bruijn probability distribution on $\mathbb{R}_{\geq 0}$, and $\varepsilon_N = \varepsilon_N(k, \alpha, f, R) \rightarrow 0$ as $N \rightarrow \infty$ satisfies

$$\varepsilon_N = O\left(\frac{\log \log N}{\log N}\right) + O\left(\frac{(\log N)^{|\alpha| - \Re \alpha}}{R^{\eta-1}}\right). \quad (4)$$

The constants implied by the O -notation in (4) depend on k , α , f , and R .

Remark 1.2. The integral in (2) is $O(1)$. However –depending on the function f – it may tend to zero as $N \rightarrow \infty$ and, if this is the case, it may be dominated by the error term ε_N . In Section 6 we discuss a concrete example where the integral term is bounded away from zero. A recent preprint by M. Avdeeva, D. Li and Ya.G. Sinai [2] gives new information on the integral term in (2) when α is a negative real number.

Remark 1.3. Our methods allow us to enlarge the set of $\alpha \in \mathbb{C}$ for which Theorem 1.1 holds, provided we modify (1) by considering the sum only over the k -free integers that are not divisible by a finite number of primes. For example, (2) holds for smooth sums over smooth *odd* k -free integers for $|\alpha| < 3$.

Theorem 1.1 shows that there is a competition between the magnitude of $R(N)$ and the regularity parameter η for the function f . Two natural choices for $R(N)$ are $\log N / \log \log N$ and $(\log N)^\tau$, $0 < \tau < 1$, and are covered by the following corollaries.

Corollary 1.4. *Let $R(N) = \frac{\log N}{\log \log N}$ in Theorem 1.1. Then*

$$\varepsilon_N = \begin{cases} O\left(\frac{\log \log N}{\log N}\right), & \text{if } \eta > |\alpha| - \Re \alpha + 2; \\ O\left(\frac{(\log \log N)^{\eta-1}}{(\log N)^{\eta-(|\alpha|-\Re \alpha+1)}}\right), & \text{if } |\alpha| - \Re \alpha < \eta \leq |\alpha| - \Re \alpha + 2. \end{cases}$$

Corollary 1.5. *Let $R(N) = (\log N)^{1-\tau}$ in Theorem 1.1. Then*

$$\varepsilon_N = \begin{cases} O\left(\frac{\log \log N}{\log N}\right), & \text{if } 0 < \tau \leq \frac{\eta - (|\alpha| - \Re \alpha + 2)}{\eta - 1} - \text{case (a)}; \\ O\left((\log N)^{-\eta(1-\tau)-\tau+(|\alpha|-\Re \alpha+1)}\right), & \text{if } 0 < \tau < \frac{\eta - (|\alpha| - \Re \alpha + 2)}{\eta - 1} < \tau < \frac{\eta - (|\alpha| - \Re \alpha + 1)}{\eta - 1} - \text{case (b)}. \end{cases}$$

The two cases (a) and (b) are summarized in Figure 1, where the trade-off between τ and η is apparent.

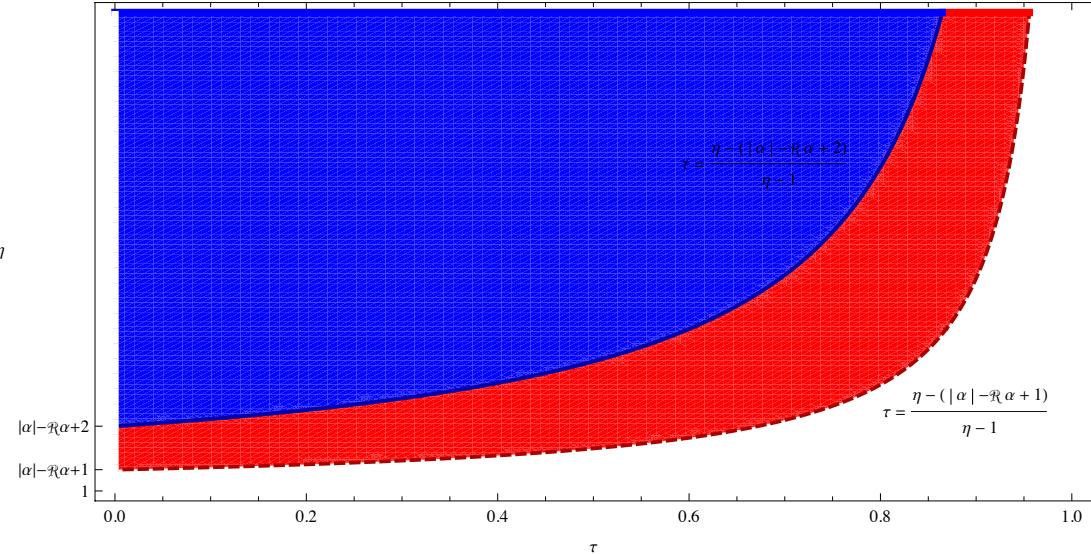


Figure 1: Plot of the two regions (a) (blue) and (b) (red), wherein the error terms $O\left(\frac{\log \log N}{\log N}\right)$ and $O\left((\log N)^{-\eta(1-\tau)-\tau+(|\alpha|-\Re \alpha+1)}\right)$ in Corollary 1.5 dominate respectively. The boundary of the two regions are hyperbolæ, whose equations are also shown.

The paper is organized as follows. Section 2 serves as context and motivation; it includes results concerning the average of certain arithmetic functions, some asymptotic results about smooth numbers, and convolutions of the Dickman-De Bruijn distribution. In Section 3 we

introduce a complex measure and we rewrite the sum (1) by means of a random¹ variable, following ideas from Statistical Mechanics already used in [6, 5]. The main results in Section 4 are Theorem 4.1, that is devoted to the pointwise approximation of the characteristic function of the above random variable, and Proposition 4.4, dealing with some integral estimates. Theorem 1.1 follows from these results. The proof of Theorem 4.1 occupies Section 5 and Appendix A. Section 6 discusses a concrete example where the function f is C^∞ with compact support.

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2 Motivation

2.1 Normal and Average Orders of Arithmetic Functions

Hardy and Ramanujan [18] proved that the *normal order* of $\omega(n)$ and $\Omega(n)$ is $\log \log n$, i.e. for every $\varepsilon > 0$, the set of $n \leq x$ such that the inequalities

$$(1 - \varepsilon) \log \log n \leq \omega(n) \leq (1 + \varepsilon) \log \log n$$

fail to hold has cardinality $o(x)$ as $x \rightarrow \infty$ (and the same statement is true for $\Omega(n)$ in place of $\omega(n)$). Erdős and Kac [14] improved on this result and established a Central Limit Theorem:

$$\frac{1}{x} \left| \left\{ n \leq x : a \leq \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq b \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} dt + O\left(\frac{1}{\sqrt{\log \log x}}\right). \quad (5)$$

Dirichlet proved that the *average order* of $d(n)$ is $n \log n$, i.e. $\sum_{n \leq x} d(n) \sim x \log x$. More precisely

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x),$$

where γ is Euler-Mascheroni's constant and $\Delta(x) = O(x^{1/2})$. Finding the smallest $\theta > 0$ such that $\Delta(x) = O(x^{\theta+\varepsilon})$ for every $\varepsilon > 0$ is known as the *Dirichlet divisor problem*. Several authors have improved Dirichlet's bound $\theta \leq \frac{1}{2}$ towards the conjectured value $\theta = \frac{1}{4}$, the current record being $\theta \leq \frac{131}{416}$ (Huxley [24]).

¹We borrow the classical probabilistic terminology, although for most values of the parameter α , we are not dealing with a probability measure.

Mertens [27, 26] proved that the average order of $2^{\omega(n)}$ is $\frac{1}{\zeta(2)}n \log n$. More precisely

$$\sum_{n \leq x} 2^{\omega(n)} = \frac{1}{\zeta(2)}x \log x + \left(\frac{2\gamma - 1}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)} \right) x + \Delta^{(2)}(x), \quad (6)$$

where $\Delta^{(2)}(x) = O(x^{1/2} \log x)$. It is also conjectured that $\Delta^{(2)}(x) = O(x^{\theta+\varepsilon})$ with $\theta = \frac{1}{4}$, and the best known result (assuming the Riemann Hypothesis) is $\theta \leq \frac{4}{11}$ (Baker [3]).

Grosswald [16] proved that the average order of $2^{\Omega(n)}$ is $An \log^2 n$, where $A \approx 0.27317$ is an explicit constant. More precisely

$$\sum_{n \leq x} 2^{\Omega(n)} = Ax \log^2 x + Bx \log x + O(x), \quad (7)$$

where B is another explicit constant, and the error term is optimal (Bateman [4]).

It is interesting to generalize the sums (6) and (7) by replacing 2 by an arbitrary complex number z . Moreover, one can restrict the summation to the set of square-free integers (characterized by the condition $\omega(n) = \Omega(n)$) by multiplying the summand by $\mu^2(n)$, where μ is the Möbius function. Selberg [28] proved that for every $z \in \mathbb{C}$, as $x \rightarrow \infty$

$$\sum_{n \leq x} z^{\omega(n)} = F(z) x(\log x)^{z-1} + O(x(\log x)^{\Re z - 2}), \quad (8)$$

$$\sum_{n \leq x} \mu^2(n) z^{\omega(n)} = G(z) x(\log x)^{z-1} + O(x(\log x)^{\Re z - 2}), \quad (9)$$

and, under the assumption that $|z| < 2$,

$$\sum_{n \leq x} z^{\Omega(n)} = H(z) x(\log x)^{z-1} + O(x(\log x)^{\Re z - 2}), \quad (10)$$

where

$$\begin{aligned} F(z) &= \frac{1}{\Gamma(z)} \prod_p \left(1 + \frac{z}{p-1}\right) \left(1 - \frac{1}{p}\right)^z, \\ G(z) &= \frac{1}{\Gamma(z)} \prod_p \left(1 + \frac{z}{p}\right) \left(1 - \frac{1}{p}\right)^z, \\ H(z) &= \frac{1}{\Gamma(z)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z. \end{aligned} \quad (11)$$

The convergence in (8-10) with respect to z is uniform on compact sets. Notice that (8) yields the first two terms in (6), and that the condition $|z| < 2$ in (10) can not be relaxed since, for example, for $z = 2$, (10) and (7) are different. These results were further improved and generalized by Delange [11]. He proved the following

Theorem 2.1. Let f be a non-negative, integer-valued, additive function such that $f(p) = 1$, and let χ be a bounded, multiplicative function such that $\chi(p) = 1$. For $\varrho \geq 0$ let

$$\sigma_0(\varrho) = \inf \left\{ \sigma > \frac{1}{2} : \sum_{k \geq 2} \sum_p \frac{|\chi(p^k)| \varrho^{f(p^k)}}{p^{k\sigma}} < \infty \right\}$$

(let us set $\inf \emptyset = +\infty$). Let $E = \{\varrho \geq 0 : \sigma_0(\varrho) < 1\}$ and set $R = \sup E \geq 1$. Then there exists a sequence $\{A_j(z)\}_{j \geq 0}$ of holomorphic functions on $|z| < R$ (and continuous on $|z| \leq R$ if $R \in E$) such that for every $q \geq 0$

$$\sum_{n \leq x} \chi(n) z^{f(n)} = x(\log x)^{z-1} \left(\sum_{j=0}^q \frac{A_j(z)}{(\log x)^j} + O\left(\frac{1}{(\log x)^{q+1}}\right) \right) \quad \text{as } x \rightarrow \infty. \quad (12)$$

The constant implied by the O -notation is uniform in z on compact sets.

In other words, one can write an asymptotic expansion for the sum in (12) in powers of $\log x$ to arbitrary order. The functions $A_j(z)$ in (12) can be constructed explicitly, in particular

$$A_0(z) = \frac{1}{\Gamma(z)} \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{\chi(p^k) z^{f(p^k)}}{p^k} \right) \left(1 - \frac{1}{p} \right)^z.$$

Notice that the results by Selberg follow from Theorem 2.1, namely $(f, \chi) = (\omega, 1)$ gives (8), $(f, \chi) = (\omega, \mu^2)$ gives (9), and $(f, \chi) = (\Omega, 1)$ gives (10). Theorem 2.1 implies a general result concerning the average order of χ along the level sets of f :

Theorem 2.2. Let f, χ be as in Theorem 2.1. For every $m \geq 1$, there exists a sequence $\{P_j\}_{j \geq 0}$ of polynomials of degree $\leq m-1$ such that for every $q \geq 0$

$$\sum_{\substack{n \leq x \\ f(n) = m}} \chi(n) = \sum_{j=1}^q \frac{x P_j(\log \log x)}{(\log x)^{j+1}} + O\left(\frac{x(\log \log x)^{m-1}}{(\log x)^{q+2}}\right) \quad \text{as } x \rightarrow \infty.$$

The coefficient of the monomial of degree $m-1$ in P_j is

$$\frac{(-1)^j}{(m-1)!} \left. \frac{d^j}{ds^j} \left(\frac{1}{s} \prod_p \left(1 + \sum_{f(p^k)=0} \frac{\chi(p^k)}{p^{ks}} \right) \right) \right|_{s=1}.$$

In particular Theorem 2.2 implies the results by Landau [25]: as $x \rightarrow \infty$

$$\sum_{\substack{n \leq x \\ \omega(n) = m}} 1 \sim \sum_{\substack{n \leq x \\ \Omega(n) = m}} 1 \sim \sum_{\substack{n \leq x \\ \omega(n) = \Omega(n) = m}} 1 \sim \frac{x}{\log x} \frac{(\log \log x)^{m-1}}{(m-1)!}$$

by taking $(f, \chi) = (\omega, 1)$, $(f, \chi) = (\Omega, 1)$, and $(f, \chi) = (\omega, \mu^2)$ respectively. Let us also point out that the error terms coming from Theorem 2.2 (i.e. $O\left(\frac{x(\log \log x)^{m-1}}{(\log x)^2}\right)$) are better than the ones given by Landau (i.e. $O\left(\frac{x(\log \log x)^{m-2}}{\log x}\right)$). Recall that the integers for which $\Omega(n) = m$ are called *m-almost primes*.

2.1.1 Ordinary and Logarithmic Mean Values

Given an arithmetic function g , one can define the *(ordinary) mean value of g* as $M[g] = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} g(n)$, and the *logarithmic mean value of g* as $L[g] = \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{g(n)}{n}$. It is well-known that if $M[g]$ exists, then $L[g]$ also exists and they are equal. The converse is in general not true. For example, $M[\mu] = L[\mu] = 0$ and $M[\mu^2] = L[\mu^2] = \frac{6}{\pi^2}$.

The sums (1) we are concerned with can be seen as partial sums for some weighted logarithmic averages.

2.2 Sums over Smooth Integers

Sums of type (8-12) can be further generalized by setting a constraint on the size of primes in the factorization of n . Integers whose prime factors are all $\leq y$ are called *y -smooth*, while those whose prime factors are $\geq c$ are called *c -rough* or *c -jagged*. Generalizing (10), we can introduce the sum

$$\Psi(z; x, y, c) = \sum_{\substack{n \leq x \\ p|n \Rightarrow c < p \leq y}} z^{\Omega(n)}. \quad (13)$$

Clearly, if $c < 2$ we set no restriction on the roughness of the integers in the sum, and if $y \geq x$ we set no restriction for their smoothness. The inequality $|z| < 2$ in (10) can be replaced by $|z| < c$ for $c \geq 2$, and (using the notation (13) we just introduced) we have

$$\Psi(z; x, x, c) = H_c(z) x(\log x)^{z-1} + O(x(\log x)^{\Re z - 2}) \quad \text{as } x \rightarrow \infty, \quad (14)$$

where

$$H_c(z) = \frac{1}{\Gamma(z)} \prod_{p \leq c} \left(1 - \frac{1}{p}\right)^z \prod_{p > c} \left(1 - \frac{z}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^z, \quad \text{cfr. (11).} \quad (15)$$

Moreover, the results by Delange (e.g. Theorem 2.1) allow to write the asymptotic of $\Psi(z; x, x, c)$ in powers of $\log x$, where the functions $z \mapsto A_j(z)$ (see (12)) are holomorphic for $|z| < p'$, where $p' = p'(c)$ is the least prime larger than c .

For $z = 1$ we have the well-known counting of y -smooth integers:

$$\Psi(x, y) = \Psi(1; x, y, 1) = |\{n \leq x : n \text{ is } y\text{-smooth}\}|. \quad (16)$$

Dickman [12] proved that $\Psi(x, y) \sim x\rho(u)$ as $x \rightarrow \infty$ when for $y = x^{1/u}$ for some $u \geq 1$, where ρ is the Dickman-de Bruijn function, i.e. the solution of the delay differential equation $u\rho'(u) + \rho(u-1) = 0$ with initial condition $\rho(u) = 1$ for $0 \leq u \leq 1$. The range of u 's for which the above asymptotic result is valid has been significantly enlarged, and explicit error terms are known. Namely

$$\Psi(x, y) = x\rho(u) \left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right), \quad \text{where } y = x^{1/u}, \quad (17)$$

and $y > \exp((\log x)^{5/8+\varepsilon})$ (de Bruijn [7, 8, 9]), or $y > \exp((\log \log x)^{5/3+\varepsilon})$ (Hildebrand [21]). Moreover, (17) holds uniformly for $y \geq (\log x)^{2+\varepsilon}$ if and only if the Riemann Hypothesis is true (Hildebrand [21]). For smaller values of y the asymptotic result (17) is not true anymore. One has, for example,

$$\Psi(x, \log^A x) = x^{1-1/A+O(1/\log \log x)} \quad \text{for } A > 1$$

(Granville [15], p. 291) and

$$\log \Psi(x, \kappa \log x) = (\log(1 + \kappa) + \kappa \log(1 + 1/\kappa)) \frac{\log x}{\log \log x} \left(1 + O\left(\frac{1}{\log \log x}\right)\right)$$

(Granville [15], Erdős [13]).

One can further restrict the sums (13-16) to square-free integers (for which $\omega(n) = \Omega(n)$) (cfr. (9)). If y is not too small compared to x (namely $\log y / \log \log x \rightarrow \infty$), then the rhs of (16) is simply multiplied by the density of square-free numbers, i.e. $6/\pi^2$. In the case we shall consider this will not be the case, since we will deal with $y \sim \log n$.

It is also worthwhile to mention that Alladi [1], Hensley [20], and Hildebrand [22, 23] proved an analog of the Erdős-Kac result (5) for y -smooth integers, with the same mean and variance ($\sim \log \log x$) for $u = \log x / \log y = o(\log \log x)$, while the mean is $\sim u$ and the variance is $\sim u/(\log u)^2$ whenever $\log x \ll y \ll \exp((\log x)^{1/21})$.

A large amount of work on averages of multiplicative functions over smooth integers has been done by G. Tenenbaum and J. Wu [30, 31, 32] and G. Hanrot, Tenenbaum and Wu [17]. The reader can refer to [30] for an historical introduction to the subject.

2.3 Convolutions of the Dickman-de Bruijn Distribution

The case of general $\Psi(z, x, y, c)$ has been studied by DeKoninck and Hensley [10]. They proved an approximation of Ψ by means of another function ψ which, in turn, is close to $x(\log x)^{z-1} \rho_z(u)$, where ρ_z is a close relative of the Dickman-de Bruijn function.

Let us consider $\alpha \geq 1$ and let us use the notation $\Psi_\alpha(x, y) = \Psi(\alpha; x, y, \alpha)$. The asymptotic of $\Psi_\alpha(x, x)$ follows from Theorem 2.1, see (14-15). The first result for y -smooth numbers is due to Hensley [19], who gave the asymptotic of $\Psi_\alpha(x, x^{1/u})$, similarly to (17). He proved that for every $\alpha \geq 1$ and every $0 < \varepsilon < 1$,

$$\Psi_\alpha(x, x^{1/u}) = \rho_\alpha(u) x(\log x)^{\alpha-1} (1 + o(1)),$$

uniformly in $1 \leq u \leq (1 - \varepsilon) \log \log x / \log \log \log x$ as $x \rightarrow \infty$, where $\rho_\alpha(u)$ satisfies the delay differential equation involving the function A_0 (see (13)) from Theorem 2.1.

$$\begin{cases} \rho_\alpha(u) = 0 & u < 0 \\ \rho_\alpha(u) = A_0(\alpha) & 0 \leq u \leq 1 \\ -u^\alpha \rho'_\alpha(u) = \alpha(u-1)^{\alpha-1} \rho_\alpha(u-1) & u > 1. \end{cases} \quad (18)$$

For $\alpha = 1$ the function (18) agrees with the Dickman-de Bruijn function (see Section 2.2). It is convenient to introduce a probability distribution on $\mathbb{R}_{\geq 0}$, whose density $w_\alpha(u)$ satisfies,

$$w_\alpha(u) = \frac{e^{\alpha \gamma}}{\Gamma(\alpha)} u^{\alpha-1} \rho_\alpha(u).$$

Notice that for $\alpha = 1$ the two functions $\rho(u)$ and $w_1(u)$ differ by a multiplicative constant, but for general α this is not the case. The density $w_\alpha(u)$ decays faster than exponentially as $u \rightarrow \infty$ (see [7]). It is known that the characteristic function $\varphi^{(\alpha)}$ of w_α is given by

$$\varphi^{(\alpha)}(\lambda) = \int_0^\infty e^{i\lambda u} w_\alpha(u) du = \exp \left(\alpha \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv \right). \quad (19)$$

In other words, since $\varphi^{(\alpha)}(\lambda) = (\varphi^{(1)}(\lambda))^\alpha$, w_α is the density of the α -convolution of the *Dickman-de Bruijn distribution*. Later we shall use the fact that

$$\varphi^{(1)}(\lambda) \sim \frac{ie^{-\gamma}}{\lambda} \quad \text{as } |\lambda| \rightarrow \infty \quad (20)$$

In our analysis, we shall consider $\alpha \in \mathbb{C}$. In this case α -convolutions of the Dickman-de Bruijn distribution cannot be considered as probability distributions on $\mathbb{R}_{\geq 0}$, but only as *distributions in the sense of Schwartz*. More precisely, they will be elements of $(\mathcal{S}_\eta(\mathbb{R}))^*$ for $\eta > |\alpha| - \Re\alpha + 1$.

3 Reformulaton of the Problem. The Ensemble $\mathfrak{X}_N^{(k)}$

Recall the prime counting function $\pi(N) = |\{p \in \mathcal{P} : p \leq N\}|$. The classical Prime Number Theorem gives $\pi(N) \sim \frac{N}{\log N}$ as $N \rightarrow \infty$. Let us consider the set

$$\mathfrak{X}_N^{(k)} = \left\{ x = \prod_{p \leq N} p^{\nu(p)} : 0 \leq \nu(p) \leq k-1 \right\}.$$

Notice that $|\mathfrak{X}_N^{(k)}| = k^{\pi(N)}$ and $\max \mathfrak{X}_N^{(k)} = \prod_{p \leq N} p^{k-1} = e^{(k-1)\pi(N) \log \pi(N)(1+o(1))}$. This means that $\mathfrak{X}_N^{(k)}$ is a sparse set. For $x \in \mathfrak{X}_N^{(k)}$ we have $\Omega(x) = \sum_{p \leq N} \nu(p)$ and $\omega(x) = |\{p \leq N : \nu(p) > 0\}|$. By definition, all $x \in \mathfrak{X}_N^{(k)}$ are k -free, and it is easy to check that all k -free $x \leq p_{\pi(N)}$ belong to $\mathfrak{X}_N^{(k)}$.

Let us introduce a complex measure $P_\Omega^{(\alpha)}$ on $\mathfrak{X}_N^{(k)}$: for every $X \subseteq \mathfrak{X}_N^{(k)}$, let

$$P_\Omega^{(\alpha)}(X) = \sum_{x \in X} \frac{\alpha^{\Omega(x)}}{x}.$$

For example $\mathfrak{X}_5^{(2)} = \{1, 2, 3, 5, 6, 10, 15, 30\}$, and $P_\Omega^{(\alpha)}(\{1, 3, 10\}) = \frac{\alpha^0}{1} + \frac{\alpha^1}{3} + \frac{\alpha^2}{10}$. Another example is $\mathfrak{X}_4^{(3)} = \{1, 2, 3, 4, 6, 9, 18, 36\}$, where $P_\Omega^{(\alpha)}(\{1, 9, 18\}) = \frac{\alpha^0}{1} + \frac{\alpha^2}{9} + \frac{\alpha^3}{18}$.

Using the terminology of Statistical Mechanics, we introduce the *partition function*

$$Z_{\Omega, N}^{(k, \alpha)} = P_\Omega^{(\alpha)}(\mathfrak{X}_N^{(k)}).$$

We prove an asymptotic result for $Z_{\Omega, N}^{(k, \alpha)}$ as $N \rightarrow \infty$. In the case of $\Re\alpha > 0$ we have that $|Z_N^{(k, \alpha)}| \rightarrow \infty$ as $N \rightarrow \infty$ and the analogy with Statistical Mechanics suggest the term

“thermodynamical limit”. Let us define a set for the parameter α for which the partition function vanishes for sufficiently large N . This set is responsible for the restriction $|\alpha| < 2$ in our Theorem 1.1. Let

$$\mathcal{A}_\Omega^{(k)} = \{z \in \mathbb{C} : z = pe^{2\pi il/k}, p \in \mathcal{P}, 1 \leq l \leq k-1\}$$

Notice that $-\mathcal{P} \subseteq \mathcal{A}_\Omega^{(k)}$ if k is even and $\mathcal{A}_\Omega^{(2)} = -\mathcal{P}$. We have the following

Lemma 3.1. *There exist a constants $C_\Omega = C_\Omega(k, \alpha) \in \mathbb{C}$ such that, as $N \rightarrow \infty$,*

$$Z_{\Omega, N}^{(k, \alpha)} = \begin{cases} 0, & \alpha \in \mathcal{A}_\Omega^{(k)}; \\ C_\Omega (\log N)^\alpha \left(1 + O\left(\frac{1}{\log N}\right)\right), & \text{otherwise}; \end{cases} \quad (21)$$

where the constant implied by the O -notation depends on k and α .

Proof. We can write

$$Z_{\Omega, N}^{(k, \alpha)} = \prod_{p \leq N} \left(1 + \frac{\alpha}{p} + \frac{\alpha^2}{p^2} + \dots + \frac{\alpha^{k-1}}{p^{k-1}}\right). \quad (22)$$

Notice that $\sum_{l=0}^{k-1} (\alpha/u)^l = 0$ if and only if $u = \alpha e^{2pi \frac{l}{k}}$, $l = 1, \dots, k-1$, and there is a prime p of this form if and only if $\alpha \in \mathcal{A}_\Omega^{(k)}$.

Now, let $\alpha \notin \mathcal{A}_\Omega^{(k)}$, $\alpha \neq 0$. Then there exist a constant $d = d(\alpha)$ such that $z(p) = \sum_{l=0}^{k-1} (\alpha/p)^l \in B_1(1/2) = \{z \in \mathbb{C} : |z - 1| < 1/2\}$ for every $p > d$. For all such p 's we can write $\log z(p) = \log |z(p)| + i \arg(z(p))$ and choose the same branch of the logarithm. From (22) we get

$$Z_{\Omega, N}^{(k, \alpha)} = C_1 \cdot \prod_{d < p \leq N} \left(1 + \frac{\alpha}{p}\right) \cdot \prod_{d < p \leq N} \left(1 + \frac{\alpha^2/p^2 + \dots + \alpha^{k-1}/p^{k-1}}{1 + \alpha/p}\right), \quad (23)$$

where $C_1 = C_1(\alpha) = \prod_{p \leq d} z(p) \neq 0$. The second factor in (23) gives the asymptotic $(\log N)^\alpha$. In fact, by taking the logarithm, we have

$$\begin{aligned} \log Z_{\Omega, N}^{(k, \alpha)} &= \log C_1 + \alpha \sum_{d < p \leq N} \frac{1}{p} - \sum_{d < p \leq N} \left(\log \left(1 + \frac{\alpha}{p}\right) - \frac{\alpha}{p} \right) + \sum_{d < p \leq N} \log \left(1 + \frac{\alpha^2/p^2 - \alpha^k/p^k}{1 - \alpha^2/p^2}\right) = \\ &= \log C_1 + \alpha(\log \log N + C_2) + C_3 + C_4 + O\left(\frac{1}{\log N}\right), \end{aligned}$$

where $C_2 = C_2(\alpha)$, $C_3 = C_3(\alpha)$ and $C_4 = C_4(k, \alpha)$ do not depend on N . We then see immediately that (21) holds with $C_\Omega(k, \alpha) = C_1 e^{\alpha C_2 + C_3 + C_4}$. \square

Remark 3.2. A classical theorem by Mertens [27] gives $C_\Omega(2, 1) = e^{-\gamma}$.

Let us observe that the sum (1) can be written as

$$S_{\Omega,f}(k, \alpha; N) = \sum_{x \in \mathfrak{X}_N^{(k)}} f\left(\frac{\log x}{\log N}\right) P_{\Omega}^{(\alpha)}(\{x\}).$$

We can write

$$\log x = \sum_{p \leq N} \nu(p) \log p$$

and introduce the function

$$\mathfrak{X}_N^{(k)} \ni x \mapsto \xi_N(x) = \frac{\log x}{\log N} = \sum_{p \leq N} \nu(p) \frac{\log p}{\log N}.$$

Our main theorem will follow from the study of the distribution of ξ_N with respect to the measures $P_{\Omega}^{(\alpha)}$. It is convenient for us to introduce the *normalized* measure $P_{\Omega,N}^{(k,\alpha)}$ on $\mathfrak{X}_N^{(k)}$, i.e.

$$P_{\Omega,N}^{(\alpha,k)} = (Z_{\Omega,N}^{(k,\alpha)})^{-1} P_{\Omega}^{(\alpha)}.$$

For every $p \leq N$, the measure $P_{\Omega,N}^{(k,\alpha)}$ on $\mathfrak{X}_N^{(k)}$ induce a measure on $\{0, 1, \dots, k-1\}$ via the function

$$\mathfrak{X}_N^{(k)} \ni x \mapsto \nu(p) = \nu(p, x) = \max\{l \geq 0 : p^l | x\}. \quad (24)$$

We shall use the distributions of the $\nu(p)$'s to compute the ones of ξ_N . We have the following simple

Lemma 3.3. *For every $0 \leq t \leq k-1$*

$$P_{\Omega,N}^{(k,\alpha)}(\{\nu(p) = t\}) = \frac{\alpha^t p^{k-1-t} (p - \alpha)}{p^k - \alpha^k}.$$

Proof. The result follows from the straightforward computation

$$\begin{aligned} P_{\Omega,N}^{(k,\alpha)}(\{\nu(p) = t\}) &= \frac{1}{Z_{\Omega,N}^{(k,\alpha)}} \frac{\alpha^t}{p^t} \prod_{p' \leq N, p' \neq p} \left(1 + \frac{\alpha}{p'} + \dots + \frac{\alpha^{k-1}}{p'^{k-1}}\right) = \frac{\alpha^t / p^t}{1 + \alpha/p + \dots + \alpha^{k-1}/p^{k-1}} = \\ &= \frac{\alpha^t}{p^t} \frac{1 - \alpha/p}{1 - \alpha^k/p^k} = \frac{\alpha^t p^{k-1-t} (p - \alpha)}{p^k - \alpha^k}, \end{aligned}$$

and the normalization of the measure $P_{\Omega,N}^{(k,\alpha)}$. □

Remark 3.4. The functions $\nu(p)$ are independent with respect to the measure $P_{\Omega,N}^{(k,\alpha)}$, i.e. for every $r \geq 1$, every $p_{j_1} < p_{j_2} < \dots < p_{j_r} \leq N$, and every $(\epsilon_1, \epsilon_2, \dots, \epsilon_r) \in \{0, 1, \dots, k-1\}^r$, we have

$$P_{\Omega,N}^{(k,\alpha)}(\{\nu(p_{j_1}) = \epsilon_1, \nu(p_{j_2}) = \epsilon_2, \dots, \nu(p_{j_r}) = \epsilon_r\}) = \prod_{l=1}^r P_{\Omega,N}^{(k,\alpha)}(\{\nu(p_{j_l}) = \epsilon_l\}).$$

On the other hand, by Lemma 3.3, the $\nu(p)$'s are not identically distributed with respect to $P_{\Omega,N}^{(k,\alpha)}$. Notice, for instance, that $P_{\Omega,N}^{(k,\alpha)}(\{\nu(p_{\pi(N)}) = 0\}) = 1 - \frac{\alpha}{N} + O\left(\frac{1}{N^k}\right)$ as $N \rightarrow \infty$. The dependence of the “probabilities” $P_{\Omega,N}^{(k,\alpha)}(\{\nu(p) = t\})$ upon N will play a crucial role in the proof of our main theorem.

Let us define the rational functions $F_{\Omega,t}^{(k,\alpha)} \in \mathbb{C}(u)$ such that $F_{\Omega,t}^{(k,\alpha)}(p) = P_{\Omega,N}^{(k,\alpha)}(\{\nu(p) = t\})$:

$$F_{\Omega,t}^{(k,\alpha)}(u) = \frac{\alpha^t u^{k-1-t}(u - \alpha)}{u^k - \alpha^k} = \frac{\alpha^t u^{k-1-t}}{u^{k-1} + \alpha u^{k-2} + \alpha^2 u^{k-3} + \dots + \alpha^{k-2} u + \alpha^{k-1}}, \quad (25)$$

where $0 \leq t \leq k-1$. Notice that the poles of $F_{\Omega,t}^{(k,\alpha)}(u)$ are $-\alpha e^{2\pi i \frac{l}{k}}$, $1 \leq l \leq k-1$. Let $b_{\Omega,t}^{(k,\alpha)}(l)$, $l \geq 0$ the coefficient of the Laurent series for $F_{\Omega,t}^{(k,\alpha)}(u)$ on the neighborhood of infinity $|u| > |\alpha|$,

$$F_{\Omega,t}^{(k,\alpha)}(u) = \sum_{l=0}^{\infty} \frac{b_{\Omega,t}^{(k,\alpha)}(l)}{u^l}.$$

Set $b_{\Omega,0}^{(k,\alpha)}(l) = 0$ for $l < 0$. We have the following two simple lemmata:

Lemma 3.5.

$$b_{\Omega,0}^{(k,\alpha)}(l) = \begin{cases} \alpha^l, & l \equiv 0 \pmod{k}; \\ -\alpha^l, & l \equiv 1 \pmod{k}; \\ 0, & l \equiv 2, \dots, k-1 \pmod{k}. \end{cases}$$

Lemma 3.6.

$$b_{\Omega,t}^{(k,\alpha)}(l) = \alpha^t b_{\Omega,0}^{(k,\alpha)}(l-t), \quad t \geq 0.$$

4 Limit theorem for ξ_N

As commonly done in Probability Theory, weak convergence of measures is obtained by showing the pointwise convergence of the corresponding characteristic functions (inverse Fourier transforms). Let us define, for $\lambda \in \mathbb{R}$,

$$\varphi_{\Omega,N}^{(k,\alpha)}(\lambda) = \mathbb{E}_{P_{\Omega,N}^{(k,\alpha)}}(e^{i\lambda\xi_N}).$$

Notice that the chosen normalizations for $P_{\Omega,N}^{(k,\alpha)}$ gives $\varphi_{\Omega,N}^{(k,\alpha)}(0) = 1$. Moreover, when $P_{\Omega,N}^{(k,\alpha)}$ is a *probability* measure (i.e. when α is a positive real number), then we have the inequality $|\varphi_{\Omega,N}^{(k,\alpha)}(\lambda)| \leq 1$, but this is in general not true. Instead, we have

$$\begin{aligned} \left| \varphi_{\Omega,N}^{(k,\alpha)}(\lambda) \right| &\leq \mathbb{E}_{|P_{\Omega,N}^{(k,\alpha)}|} 1 = \frac{Z_{\Omega,N}^{(k,|\alpha|)}}{\left| Z_{\Omega,N}^{(k,\alpha)} \right|} = \frac{C_{\Omega}(k, |\alpha|)}{|C_{\Omega}(k, \alpha)|} \frac{(\log N)^{|\alpha|} (1 + O(1/\log N))}{(\log N)^{\Re \alpha} (1 + O(1/\log N))} = \\ &= O((\log N)^{|\alpha| - \Re \alpha}), \end{aligned} \quad (26)$$

where the constant implied by the O -notation depends only on k , and α , uniformly in λ .

Recall $\varphi^{(\alpha)}(\lambda)$ from (19), i.e. the characteristic function of the α -convolution of the Dickman-de Bruijn distribution. We prove the following

Theorem 4.1. *Let $|\alpha| < 2$, and assume that $\lambda = o(\log N)$ as $N \rightarrow \infty$. Then*

$$\varphi_{\Omega, N}^{(k, \alpha)}(\lambda) = \varphi^{(\alpha)}(\lambda) \left(1 + O\left(\frac{1}{\log N}\right) + O(\varepsilon \log |\varepsilon|) \right),$$

where $\varepsilon = \frac{\lambda}{\log N} = o(1)$ as $N \rightarrow \infty$. The constants implied by the O -notation depend only on k and α .

Remark 4.2. The case of $\lambda \rightarrow 0$ is of no harm in dealing with characteristic functions of normalized measures (recall that $\varphi_{\Omega, N}^{(k, \alpha)}(0) = 1$), and the error-term bounds obtained are the same as for $\lambda = O(1)$. The term $O\left(\frac{1}{\log N}\right)$ in the theorem above prevents an underestimate of the error term in the (uninteresting) case when $\lambda \rightarrow 0$, i.e. $\varepsilon = o\left(\frac{1}{\log N}\right)$. The interesting application of the above result is for $|\lambda| \rightarrow \infty$ more slowly than $\log N$, and the corresponding error term can be simply written as $O(\varepsilon \log |\varepsilon|)$.

Remark 4.3. Theorem 4.1 generalizes the main result in [6, 5] (where the case $(k, \alpha) = (2, 1)$ is addressed), and provides an explicit error term (as opposed to simply an error term $o(1)$).

A consequence of Theorem 4.1 is the following

Proposition 4.4. *Let us fix $k \geq 2$, $|\alpha| < 2$, $f \in \mathcal{S}_\eta$ with $\eta \geq |\alpha| - \Re \alpha + 1$. Then for every $R = R(N)$ such that*

$$\frac{R}{\log N} \rightarrow 0 \quad \text{and} \quad \frac{(\log N)^{|\alpha| - \Re \alpha}}{R^{\eta-1}} \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

$$\int_{\mathbb{R}} \varphi_{\Omega, N}^{(k, \alpha)}(\lambda) \hat{f}(\lambda) d\lambda = \int_{|\lambda| \leq R} \varphi^{(\alpha)}(\lambda) \hat{f}(\lambda) d\lambda + \varepsilon_N,$$

where $\varepsilon_N = \varepsilon_N(k, \alpha, f, R) \rightarrow 0$ as $N \rightarrow \infty$ satisfies

$$\varepsilon_N = O\left(\frac{\log \log N}{\log N}\right) + O\left(\frac{(\log N)^{|\alpha| - \Re \alpha}}{R^{\eta-1}}\right).$$

Proof. For every $R = R(N)$ such that $R(N) = o(\log N)$ as $N \rightarrow \infty$, let us write

$$\int_{\mathbb{R}} \varphi_{\Omega, N}^{(k, \alpha)}(\lambda) \hat{f}(\lambda) d\lambda = \int_{|\lambda| \leq R} + \int_{|\lambda| > R} = \mathcal{I}_1 + \mathcal{I}_2.$$

Now we can use use Theorem 4.1:

$$\begin{aligned}\mathcal{I}_1 &= \int_{|\lambda| \leq R} \varphi^{(\alpha)}(\lambda) \hat{f}(\lambda) d\lambda + O\left(\int_{|\lambda| < R} \varphi^{(\alpha)}(\lambda) \frac{\lambda}{\log N} \log \left| \frac{\lambda}{\log N} \right| \hat{f}(\lambda) d\lambda\right) + \\ &\quad + O\left(\int_{|\lambda| \leq R} \varphi^{(\alpha)}(\lambda) \frac{1}{\log N} \hat{f}(\lambda) d\lambda\right) = \mathcal{I}_{1,1} + \mathcal{I}_{1,2} + \mathcal{I}_{1,3}.\end{aligned}$$

Notice that, by (20), we have $\varphi^{(\alpha)}(\lambda) = O(\lambda^{-\Re\alpha})$ as $\lambda \rightarrow \infty$. We can write

$$\begin{aligned}\mathcal{I}_{1,2} &= O\left(\int_{|\lambda| \leq 1} \left| \frac{\lambda}{\log N} \log \left| \frac{\lambda}{\log N} \right| \right| d\lambda\right) + O\left(\int_{1 < |\lambda| \leq R} \frac{\lambda^{1-\Re\alpha-\eta}}{\log N} \log \left| \frac{\lambda}{\log N} \right| d\lambda\right) = \\ &= O\left(\frac{\log \log N}{\log N}\right) + \begin{cases} O\left(\frac{\log R}{\log N} \log\left(\frac{\log^2 N}{R}\right)\right), & \Re\alpha + \eta = 2; \\ O\left(\frac{\log \log N}{\log N}\right) + O\left(\frac{R^{2-\Re\alpha-\eta}}{\log N} \log\left(\frac{\log N}{R}\right)\right), & \text{otherwise}; \end{cases} \\ \mathcal{I}_{1,3} &= O\left(\frac{1}{\log N}\right) + O\left(\int_{1 < \lambda \leq R} \frac{\lambda^{-\Re\alpha-\eta}}{\log N} d\lambda\right) = \begin{cases} O\left(\frac{\log R}{\log N}\right), & \Re\alpha + \eta = 1; \\ O\left(\frac{R^{1-\Re\alpha-\eta}}{\log N}\right), & \text{otherwise}. \end{cases}\end{aligned}$$

Let us observe that $\mathcal{I}_{1,3} = O(\mathcal{I}_{1,2})$. To estimate \mathcal{I}_2 we use the trivial estimate (26) and the fact that $\eta > 1$:

$$\mathcal{I}_2 = O\left((\log N)^{|\alpha|-\Re\alpha} \int_{\lambda > R} \frac{d\lambda}{\lambda^\eta}\right) = O\left(\frac{(\log N)^{|\alpha|-\Re\alpha}}{R^{\eta-1}}\right).$$

Summarizing,

$$\int_{\mathbb{R}} \varphi_{\Omega, N}^{(k, \alpha)}(\lambda) \hat{f}(\lambda) d\lambda = \int_{|\lambda| \leq R} \varphi^{(\alpha)}(\lambda) \hat{f}(\lambda) d\lambda + O\left(\frac{\log \log N}{\log N}\right) + O\left(\frac{(\log N)^{|\alpha|-\Re\alpha}}{R^{\eta-1}}\right) + \quad (27)$$

$$+ \begin{cases} O\left(\frac{\log R}{\log N} \log\left(\frac{\log^2 N}{R}\right)\right), & \Re\alpha + \eta = 2; \\ O\left(\frac{R^{2-\Re\alpha-\eta}}{\log N} \log\left(\frac{\log N}{R}\right)\right), & \text{otherwise}. \end{cases} \quad (28)$$

One can check that the term in (28) is dominated by those in (27). \square

Theorem 1.1 follows now immediately. In fact,

$$S_{\Omega, f}(k, \alpha; N) = Z_{\Omega, N}^{(k, \alpha)} \int_{\mathbb{R}} \varphi_{\Omega, N}^{(k, \alpha)}(\lambda) \hat{f}(\lambda) d\lambda,$$

and Lemma 3.1 and Proposition 4.4 give the desired statements (2)-(4).

5 Proof of Theorem 4.1

We shall need the following result

Lemma 5.1. *Let*

$$\mathcal{E}_\Omega^{(k,\alpha)}(u, \lambda, N) = \frac{d}{du} F_{\Omega,0}^{(k,\alpha)}(u) - \frac{\alpha}{u^2} + \left(\frac{d}{du} F_{\Omega,1}^{(k,\alpha)}(u) + \frac{\alpha}{u^2} \right) e^{i\lambda \frac{\log u}{\log N}} + \sum_{t=2}^{k-1} \frac{d}{du} F_{\Omega,t}^{(k,\alpha)}(u) e^{i\lambda t \frac{\log u}{\log N}},$$

where $|u| > |\alpha|$. Then

$$\mathcal{E}_\Omega^{(k,\alpha)}(u, \lambda, N) = \sum_{j=1}^{k-1} \frac{\alpha^j \left(e^{i\lambda j \frac{\log u}{\log N}} - 1 \right)}{u^{j+2}} O(1),$$

where the constant implied by the O -notation depends on j , k and α .

Proof. We have

$$\begin{aligned} \mathcal{E}_\Omega^{(k,\alpha)}(u, \lambda, N) &= \\ &= \sum_{l=2}^{\infty} \frac{-l b_{\Omega,0}^{(k,\alpha)}(l)}{u^{l+1}} + e^{i\lambda \frac{\log u}{\log N}} \sum_{l=2}^{\infty} \frac{-l\alpha b_{\Omega,0}^{(k,\alpha)}(l-1)}{u^{l+1}} + \sum_{t=2}^{k-1} e^{i\lambda t \frac{\log u}{\log N}} \sum_{l=t+1}^{\infty} \frac{-l\alpha^t b_{\Omega,0}^{(k,\alpha)}(l-t)}{u^{l+1}} = \\ &= \sum_{l=2}^{\infty} \frac{-l}{u^{l+1}} \sum_{j=0}^{l \wedge (k-1)} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) e^{i\lambda j \frac{\log u}{\log N}} = \\ &= \sum_{l=2}^{\infty} \frac{-l}{u^{l+1}} \sum_{j=0}^{l \wedge (k-1)} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) \sum_{s=0}^{\infty} \frac{\left(i\lambda j \frac{\log u}{\log N} \right)^s}{s!}. \end{aligned} \tag{29}$$

The terms corresponding to $s = 0$ in the sum (29) can be removed, as Lemmata 3.5-3.6 yield

$$\sum_{j=0}^{l \wedge (k-1)} \alpha^j b_{\Omega,0}(l-j) = 0.$$

In fact, if $2 \leq l \leq k-1$, then $\sum_{j=0}^l \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) = \alpha^{l-1} b_{\Omega,0}^{(k,\alpha)}(1) + \alpha^l b_{\Omega,0}^{(k,\alpha)}(0) = 0$. If $l \geq k$, say $l = ck + d$, with $c \geq 1$ and $0 \leq d \leq k-1$, then

$$\sum_{j=0}^{k-1} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) = \begin{cases} b_{\Omega,0}^{(k,\alpha)}(ck) + \alpha^{k-1} b_{\Omega,0}^{(k,\alpha)}((c-1)k+1) = 0, & \text{if } d=0; \\ \alpha^{d-1} b_{\Omega,0}^{(k,\alpha)}(ck+1) + \alpha^d b_{\Omega,0}^{(k,\alpha)}(ck) = 0, & \text{if } d \geq 1. \end{cases}$$

We proceed from (29), after noticing that the terms corresponding to $j = 0$ can be removed too:

$$\begin{aligned}
\mathcal{E}_\Omega^{(k,\alpha)}(u, \lambda, N) &= \sum_{l=2}^{\infty} \frac{-l}{u^{l+1}} \sum_{j=0}^{l \wedge (k-1)} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) \sum_{s=1}^{\infty} \frac{\left(i\lambda j \frac{\log u}{\log N}\right)^s}{s!} = \\
&= \sum_{l=2}^{\infty} \frac{-l}{u^{l+1}} \sum_{j=1}^{l \wedge (k-1)} \alpha^j b_{\Omega,0}^{(k,\alpha)}(l-j) \left(e^{i\lambda j \frac{\log u}{\log N}} - 1\right) = \\
&= \sum_{j=1}^{k-1} \alpha^j \left(e^{i\lambda j \frac{\log u}{\log N}} - 1\right) \sum_{l=j+1}^{\infty} \frac{-l b_{\Omega,0}^{(k,\alpha)}(l-j)}{u^{l+1}} = \\
&= \sum_{j=1}^{k-1} \frac{\alpha^j \left(e^{i\lambda j \frac{\log u}{\log N}} - 1\right)}{u^{j+2}} O(1),
\end{aligned}$$

where the constant implied by the O -notation depends on j , k and α . \square

5.1 The main step

We can write

$$\begin{aligned}
\varphi_{\Omega,N}^{(k,\alpha)}(\lambda) &= \sum_{x \in \mathfrak{X}_N^{(k)}} \exp\left(i\lambda \sum_{p \leq N} \nu(p) \frac{\log p}{\log N}\right) P_{\Omega,N}^{(k,\alpha)}(\{x\}) = \\
&= \sum_{\epsilon(p_1), \dots, \epsilon(p_{\pi(N)}) \in \{0,1,\dots,k-1\}} \prod_{p \leq N} \exp\left(i\lambda \epsilon(p) \frac{\log p}{\log N}\right) P_{\Omega,N}^{(k,\alpha)}(\{\nu(p) = \epsilon(p)\}) = \\
&= \prod_{p \leq N} \sum_{t=0}^{k-1} e^{i\lambda t \frac{\log p}{\log N}} P_{\Omega,N}^{(k,\alpha)}(\{\nu(p) = t\}).
\end{aligned}$$

Now, by Lemma 3.3 and (25), we get

$$\begin{aligned}
\varphi_{\Omega,N}^{(k,\alpha)}(\lambda) &= \prod_{p \leq N} \left(F_{\Omega,0}^{(k,\alpha)}(p) + F_{\Omega,1}^{(k,\alpha)}(p) e^{i\lambda \frac{\log p}{\log N}} + \sum_{t=2}^{k-1} e^{i\lambda t \frac{\log p}{\log N}} F_{\Omega,t}^{(k,\alpha)}(p) \right) = \\
&= \prod_{p \leq N} \left(1 + \frac{\alpha}{p} \left(e^{i\lambda \frac{\log p}{\log N}} - 1 \right) + \sum_{t=2}^{k-1} G_{\Omega,t,N}^{(k,\alpha)}(p, \lambda) \right), \tag{30}
\end{aligned}$$

where

$$G_{\Omega,t,N}^{(k,\alpha)}(u, \lambda) = \begin{cases} F_{\Omega,0}^{(k,\alpha)}(u) - 1 + \frac{\alpha}{u} + \left(F_{\Omega,1}^{(k,\alpha)}(u) - \frac{\alpha}{u}\right) e^{i\lambda \frac{\log u}{\log N}} + F_{\Omega,2}^{(k,\alpha)}(u) e^{2i\lambda \frac{\log u}{\log N}}, & t = 2; \\ F_{\Omega,t}^{(k,\alpha)}(u) e^{i\lambda t \frac{\log u}{\log N}}, & 3 \leq t \leq k-1. \end{cases}$$

Lemmata 3.5 and 3.6 imply that $G_{\Omega,t,N}^{(k,\alpha)}(u, \lambda) = O(1/u^t)$, for $2 \leq t \leq k-1$, where the constants implied by the O -notations depend only on k and α . This means, in particular, that there exists $d_* = d_*(k, \alpha)$ such that for every $\lambda \in \mathbb{R}$, every $N \geq 2$ and every $p > d_*$, the complex numbers

$$z(p) = z^{(k,\alpha)}(p, \lambda, N) = 1 + \frac{\alpha}{p} \left(e^{i\lambda \frac{\log p}{\log N}} - 1 \right) + \sum_{t=2}^{k-1} G_{\Omega,t,N}^{(k,\alpha)}(p, \lambda)$$

belong to the open disk of radius $1/2$, centered at 1 . Thus we can write $\log z(p) = \log |z(p)| + i \arg z(p)$ and choose a branch of the logarithm consistently for all $p > d_*$. Let us also assume that $d_* > |\alpha|$ (we shall use this fact later). We get

$$\varphi_{\Omega,N}^{(k,\alpha)}(\lambda) = \prod_{p \leq d_*} z^{(k,\alpha)}(p, \lambda, N) \cdot \tilde{\varphi}_{\Omega,N}^{(k,\alpha)}(\lambda) \quad (31)$$

where, after taking the logarithm,

$$\log \tilde{\varphi}_{\Omega,N}^{(k,\alpha)}(\lambda) = \sum_{d_* < p \leq N} \log z^{(k,\alpha)}(p, \lambda, N).$$

Abel summation yields

$$\log \tilde{\varphi}_{\Omega,N}^{(k,\alpha)}(\lambda) = \pi(N) \mathcal{L}(N, \lambda, N) - \pi(d_*) \mathcal{L}(d_*, \lambda, N) - \int_{d_*}^N \pi(u) \frac{\partial}{\partial u} \mathcal{L}(u, \lambda, N) du, \quad (32)$$

where $\mathcal{L}(u, \lambda, N) = \mathcal{L}^{(k,\alpha)}(u, \lambda, N) = \log z^{(k,\alpha)}(u, \lambda, N)$.

Remark 5.2. As pointed out by A.J. Hildebrand to the author, one can try to estimate (30) using a result by Tenenbaum ([29], Chapter III.5). This approach will be the subject of future investigation. Likely, it will allow for a wider range for R in Theorem 1.1 and reduce the error term in the case of small η .

5.2 The boundary terms

Let us estimate the first two terms in the rhs of (32). By Lemmata 3.5-3.6

$$\begin{aligned} \pi(N) \mathcal{L}(N, \lambda, N) &= \pi(N) \log \left(1 + \frac{\alpha}{N} (e^{i\lambda} - 1) + \sum_{t=2}^{k-1} G_{\Omega,t,N}^{(k,\alpha)}(N, \lambda) \right) = \\ &= O\left(\frac{N}{\log N}\right) \log \left(1 + O\left(\frac{1}{N}\right) (e^{i\lambda} - 1) + O\left(\frac{1}{N^2}\right) (1 + e^{i\lambda} + e^{2i\lambda}) + \right. \\ &\quad \left. + \sum_{t=3}^{k-1} O\left(\frac{1}{N^t}\right) e^{it\lambda} \right) = O\left(\frac{1}{\log N}\right), \end{aligned} \quad (33)$$

$$\pi(d_*) \mathcal{L}(d_*, \lambda, N) = \pi(d_*) \log \left(1 + \frac{\alpha}{d_*} \left(e^{i\lambda \frac{d_*}{\log N}} - 1 \right) + O\left(\frac{1}{N^2}\right) \right) = O(\varepsilon) + O\left(\frac{1}{N^2}\right), \quad (34)$$

where the constants implied by the O -notation depend only on k and α . The two boundary terms in the rhs of (32) are therefore $O\left(\frac{1}{\log N}\right) + O(\varepsilon)$ as $N \rightarrow \infty$.

5.3 The integral term

Let us now estimate the integral in the rhs of (32). Recall the functions $G_{\Omega,t,N}^{(k,\alpha)}(u, \lambda)$ and $\mathcal{E}_{\Omega}^{(k,\alpha)}(u, \lambda, N)$ introduced above. We have

$$\begin{aligned}
\frac{\partial}{\partial u} \mathcal{L}(u, \lambda, N) &= \frac{1}{z^{(k,\alpha)}(u, \lambda, N)} \left(-\frac{\alpha}{u^2} \left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right) + \frac{i\alpha\lambda}{u^2 \log N} e^{i\lambda \frac{\log u}{\log N}} + \frac{d}{du} F_{\Omega,0}^{(k,\alpha)}(u) - \frac{\alpha}{u^2} + \right. \\
&+ \left(\frac{d}{du} F_{\Omega,1}^{(k,\alpha)}(u) + \frac{\alpha}{u^2} \right) e^{i\lambda \frac{\log u}{\log N}} + \frac{i\lambda}{u \log N} \left(F_{\Omega,1}^{(k,\alpha)}(u) - \frac{\alpha}{u} \right) e^{i\lambda \frac{\log u}{\log N}} + \\
&+ \sum_{t=2}^{k-1} \left(\frac{d}{du} F_{\Omega,t}^{(k,\alpha)}(u) e^{i\lambda t \frac{\log u}{\log N}} + \frac{i\lambda t}{u \log N} F_{\Omega,t}^{(k,\alpha)}(u) e^{i\lambda t \frac{\log u}{\log N}} \right) \Big) = \\
&= \left(1 - \frac{\frac{\alpha}{u} \left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right) + \sum_{t=2}^{k-1} G_{\Omega,t,N}^{(k,\alpha)}(u, \lambda)}{z^{(k,\alpha)}(u, \lambda, N)} \right) \left(-\frac{\alpha}{u^2} \left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right) + \frac{i\alpha\lambda}{u^2 \log N} e^{i\lambda \frac{\log u}{\log N}} + \right. \\
&\quad \left. + \mathcal{E}_{\Omega}^{(k,\alpha)}(u, \lambda, N) + \frac{\lambda}{u^3 \log N} O(1) + \sum_{t=2}^{k-1} \frac{\lambda}{u^{t+1} \log N} O(1) \right) \tag{35}
\end{aligned}$$

Since $d_* \leq u \leq N$, let us notice that $\frac{\alpha}{u} \left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right) \frac{1}{z(u, \lambda, N)} = \frac{1}{u} \left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right) O(1)$. Moreover, $\frac{1}{z(u, \lambda, N)} \sum_{t=2}^{k-1} G_{\Omega,t,N}^{(k,\alpha)}(u, \lambda) = O\left(\frac{1}{u^2}\right)$. Thus, the first bracket in (35) can be written as $I_{1,1} + I_{1,2} + I_{1,3}$, where $I_{1,j} = I_{\Omega,1,j}^{(k,\alpha)}(u, \lambda, N)$, $j = 1, 2, 3$, and

$$\begin{aligned}
I_{1,1} &= 1, \\
I_{1,2} &= \frac{O(1)}{u} \left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right), \\
I_{1,3} &= O\left(\frac{1}{u^2}\right).
\end{aligned}$$

Let us look at the second bracket in (35). By Lemma (5.1) we get $k - 1$ terms of the form $\frac{O(1)}{u^{j+2}} \left(e^{i\lambda j \frac{\log u}{\log N}} - 1 \right)$, $1 \leq j \leq k - 1$ (it would not be enough to estimate them as $O(1/u^{j+2})$ as we want a better control of error terms). The implied constants can be chosen in order to not depend upon j but only on k and α . Let us also combine the last two terms in (35) into $O\left(\frac{\varepsilon}{u^3}\right)$. This means that the the second bracket in (35) can be written as $I_{2,1} + I_{2,2} + \dots + I_{2,k+2}$, where $I_{2,j} = I_{\Omega,2,j}^{(k,\alpha)}(u, \lambda, N)$, $1 \leq j \leq k + 2$, and

$$\begin{aligned}
I_{2,1} &= -\frac{\alpha}{u^2} \left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right), \\
I_{2,2} &= \frac{i\alpha\varepsilon}{u^2} e^{i\lambda \frac{\log u}{\log N}}, \\
I_{2,j} &= \frac{O(1) \left(e^{i\lambda(j-2) \frac{\log u}{\log N}} - 1 \right)}{u^j}, \quad 3 \leq j \leq k + 1, \\
I_{2,k+2} &= O\left(\frac{\varepsilon}{u^3}\right).
\end{aligned}$$

Recall that all the constants implied by the above O -notations depend only on k, α , and not on λ, u , and N . Let us also write

$$\pi(u) = \frac{u}{\log u} + O\left(\frac{u}{\log^2 u}\right) = I_{0,1} + I_{0,2}.$$

The integral in the rhs of (32) becomes now

$$-\int_{d_*}^N \pi(u) \frac{\partial}{\partial u} \mathcal{L}(u, \lambda, N) du = -\sum_{j=1}^2 \sum_{j'=1}^3 \sum_{j''=1}^{k+2} \int_{d_*}^N I_{0,j} I_{1,j'} I_{2,j''} du. \quad (36)$$

We claim that, amongst the $6k+12$ integrals in (36), the one corresponding to $j = j' = j'' = 1$ is the main term, and the remaining $6k+11$ integrals are $O\left(\frac{1}{\log N}\right) + O(\varepsilon \log |\varepsilon|)$. Let us perform the change of variables $v = \frac{\log u}{\log N}$. We have

$$\begin{aligned} J_{1,1,1} &= -\int_{d_*}^N I_{0,1} I_{1,1} I_{2,1} du = \alpha \int_{d_*}^N \frac{e^{i\lambda \frac{\log u}{\log N}} - 1}{u \log u} du = \alpha \int_{\log d_*/\log N}^1 \frac{e^{i\lambda v} - 1}{v} dv = \\ &= \alpha \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv + O(\varepsilon). \end{aligned} \quad (37)$$

The fact that

$$J_{j,j',j''} = -\int_{d_*}^N I_{0,j} I_{1,j'} I_{2,j''} du = O\left(\frac{1}{\log N}\right) + O(\varepsilon \log |\varepsilon|) \quad (38)$$

for all other values of j, j', j'' is shown in Appendix A. By (32-34, 36-38), we have that

$$\log \tilde{\varphi}_{*,N}^{(k,\alpha)}(\lambda) = \alpha \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv + O\left(\frac{1}{\log N}\right) + O(\varepsilon \log |\varepsilon|). \quad (39)$$

The first factor in the rhs of (31) can be written as follows as $N \rightarrow \infty$

$$\begin{aligned} \prod_{p \leq d_*} z^{(k,\alpha)}(p, \lambda, N) &= \prod_{p \leq d_*} \sum_{t=0}^{k-1} e^{i\lambda \frac{\log p}{\log N}} F_{\Omega,t}^{(k,\alpha)}(p) \\ &= \prod_{p \leq d_*} \left(\sum_{t=0}^{k-1} F_{\Omega,t}^{(k,\alpha)}(p) + O(\varepsilon) \right) = 1 + O(\varepsilon), \end{aligned} \quad (40)$$

where we used the fact that

$$\sum_{t=0}^{k-1} F_{\Omega,t}^{(k,\alpha)}(p) = \frac{\sum_{t=0}^{k-1} \alpha^t p^{k-1-t}}{p^{k-1} + \alpha p^{k-2} + \alpha^2 p^{k-3} + \dots + \alpha^{k-1}} = 1.$$

Now, combining (39) and (40), we get

$$\begin{aligned} \varphi_{\Omega,N}^{(k,\alpha)}(\lambda) &= (1 + O(\varepsilon)) \exp \left(\alpha \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv + O\left(\frac{1}{\log N}\right) + O(\varepsilon \log |\varepsilon|) \right) = \\ &= \exp \left(\alpha \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv \right) \left(1 + O\left(\frac{1}{\log N}\right) + O(\varepsilon \log |\varepsilon|) \right) \end{aligned}$$

and this concludes the proof of Theorem 4.1.

6 An Example

We pointed out that, for general α and f , the integral

$$I = \int_{|\lambda| \leq R} \hat{f}(\lambda) \varphi^{(\alpha)}(\lambda) d\lambda$$

in (2) may tend to zero as $N \rightarrow \infty$ (recall: $R = R(N)$). In this section we discuss an example where $\alpha = -1$ and the above integral is bounded away from zero as $N \rightarrow \infty$. We have the following

Proposition 6.1. *Let $f(u) = \mathbf{1}_{[-1,1]} e^{-\frac{1}{1-u^2}}$. Then, for sufficiently large N ,*

$$\left| \int_{|\lambda| \leq R} \hat{f}(\lambda) \varphi^{(-1)}(\lambda) d\lambda \right| \geq \frac{3}{100}. \quad (41)$$

Proof. Since $S_{\Omega,f}(k, -1; N)$ is real, then by Theorem 1.1, $\Im I = o(1)$ as $N \rightarrow \infty$. Thus, for every $\delta > 0$, there exists N_δ such that $|\Im I| \leq \delta$ for every $N \geq N_\delta$. Let us now focus on $\Re I$. Notice that \hat{f} is real-valued and thus $\Re I = \int_{|\lambda| \leq R} \hat{f}(\lambda) \Re \varphi^{(-1)}(\lambda) d\lambda$. We can write

$$\Re \varphi^{(-1)}(\lambda) = e^{\gamma - \text{Ci}(\lambda)} \lambda \cos(\text{Si}(\lambda)) \quad (42)$$

where $\text{Ci}(\lambda) = -\int_\lambda^\infty \frac{\cos t dt}{t}$ and $\text{Si}(\lambda) = \int_0^\lambda \frac{\sin t dt}{t}$. We can use the expression (42) to obtain the estimate² $|\Re \varphi^{(-1)}(\lambda)| \leq e^\gamma$, valid for all $\lambda \in \mathbb{R}$. Let us write

$$\Re I = \int_{|\lambda| \leq R} \hat{f}(\lambda) \Re \varphi^{(-1)}(\lambda) d\lambda = \int_{|\lambda| \leq r} + \int_{r < |\lambda| \leq R} = I_1 + I_2 \quad (43)$$

²Although $|\varphi^{(-1)}(\lambda)| = O(\lambda)$ as $\lambda \rightarrow \infty$, only the imaginary part of the function $\varphi^{(-1)}$ is unbounded. This property holds true only for $\alpha = -1$.

where $r > 0$ does not depend on N and will be chosen later. The idea is that I_1 can be estimated numerically with arbitrary precision using, for example, a quadrature method for the integral. More precisely, let $F(\lambda) = \hat{f}(\lambda)\Re\varphi^{(-1)}(\lambda)$ and observe that $F(\lambda) = F(-\lambda)$, so that $\int_{|\lambda| \leq r} F(\lambda) d\lambda = 2 \int_0^r F(\lambda) d\lambda$. We have

$$\int_0^r F(\lambda) d\lambda = h \sum_{m=1}^M F\left(h\left(m - \frac{1}{2}\right)\right) + \mathcal{E}(r, M),$$

where $h = \frac{r}{M}$ and $\mathcal{E}(r, M) = \frac{rh^2}{24} F''(\rho)$ for some $0 < \rho < r$. The graphs of F , F' and F'' are shown in Figure 2. One can see that $|F''(\lambda)| \leq \frac{1}{2}$. Since F can be estimated to an arbitrary precision, we

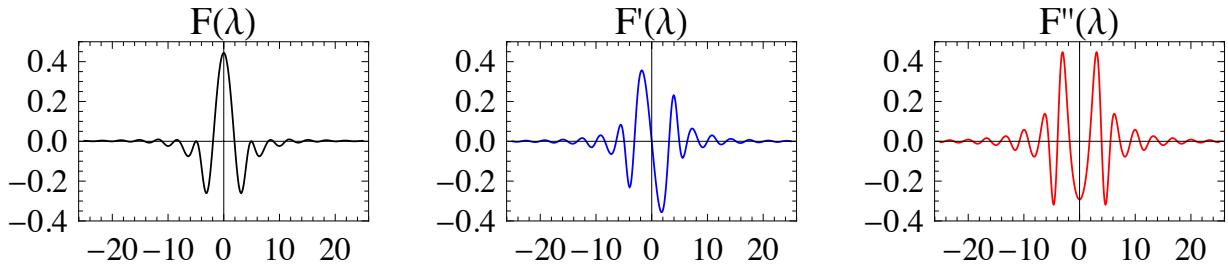


Figure 2: The graphs of F , F' and F'' .

can estimate the Riemann sum $h \sum_{m=1}^M F\left(h\left(m - \frac{1}{2}\right)\right)$ arbitrarily well for fixed r and M , and we have the estimate $|\mathcal{E}(r, M)| \leq \frac{h^2 r}{48}$. The integral I_2 in (43) can be estimated explicitly as follows:

$$\left| \int_{r < |\lambda| \leq R} F(\lambda) d\lambda \right| \leq 2 \int_r^R |F(\lambda)| d\lambda.$$

By means of stationary phase method, it can be shown that the function \hat{f} satisfies the asymptotic

$$\hat{f}(\lambda) \sim \frac{1}{\pi} \Re \left\{ \sqrt{\frac{-i\pi}{\sqrt{2i\lambda^{\frac{3}{2}}}}} e^{i\lambda - \frac{1}{4} - \sqrt{2i\lambda}} \right\}$$

as $\lambda \rightarrow \infty$. Moreover, one can check that $|\hat{f}(\lambda)| \leq \frac{3}{2\pi} e^{-\sqrt{\lambda}} \lambda^{-3/4}$ for $\lambda \geq 1$. This implies that $|F(\lambda)| \leq \frac{3e^\gamma}{2\pi} e^{-\sqrt{\lambda}} \lambda^{-3/4}$. We get

$$\left| \int_{r < |\lambda| \leq R} F(\lambda) d\lambda \right| \leq \frac{3e^\gamma}{\pi} \int_r^R e^{-\sqrt{\lambda}} \lambda^{-3/4} d\lambda = \frac{6e^\gamma}{\sqrt{\pi}} [\operatorname{erf}(\lambda^{1/4})]_{\lambda=r}^{\lambda=R} \leq \frac{6e^\gamma}{\sqrt{\pi}} \operatorname{erfc}(r^{1/4}),$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and $\text{erfc}(x) = 1 - \text{erf}(x)$. Now, let $r = 5$ and $M = 1000$. We have

$$\begin{aligned} 2h \sum_{m=1}^M F\left(h\left(m - \frac{1}{2}\right)\right) &= 0.23821680383626264857 \pm 10^{-20}, \\ \frac{h^2 r}{24} &= 5.208(3) \cdot 10^{-6}, \\ \frac{6e^\gamma}{\sqrt{\pi}} \text{erfc}(r^{1/4}) &= 0.20771652138513808389 \pm 10^{-20}, \end{aligned}$$

and therefore

$$\left| \int_{|\lambda| \leq R} F(\lambda) d\lambda \right| \geq \left| 2h \sum_{m=1}^M F\left(h\left(m - \frac{1}{2}\right)\right) \right| - |\mathcal{E}(r, M)| - \left| \int_{r < |\lambda| \leq R} F(\lambda) d\lambda \right| \geq \frac{3}{100}$$

This shows that, for sufficiently large N ,

$$\left| \int_{|\lambda| \leq R(N)} \hat{f}(\lambda) \varphi^{(-1)}(\lambda) d\lambda \right| > \frac{3}{100} + \frac{1}{2500}. \quad (44)$$

Now let us choose $\delta = \frac{1}{2500}$ so that $|\Im I| \leq \delta$ for sufficiently large N . This, together with (44), yields the desired statement (41). \square

A Estimate of the error terms

This appendix contains the estimates of the integrals $J_{j,j',j''}$ from (38) for $1 \leq j \leq 2$, $1 \leq j' \leq 3$ and $1 \leq j'' \leq k+2$, except for $(j, j', j'') = (1, 1, 1)$ already treated in (37). Recall that $\varepsilon = \frac{\lambda}{\log N}$ is assumed to tend to zero as $N \rightarrow \infty$. We will assume for simplicity that $\lambda > 0$. We group error terms in different sections, according to the methods used to estimate them.

A.1 $O(\varepsilon)$ terms

These error terms are very easy, due to the fact that $I_{2,2}$ and $I_{2,k+2}$ are $O(\varepsilon/u^2)$ and $O(\varepsilon/u^3)$ respectively and the corresponding error terms can be written as $O(\varepsilon \mathcal{I})$, where \mathcal{I} is an absolutely convergent integral. We have

$$\begin{aligned}
J_{1,2,2} &= - \int_{d_*}^N I_{0,1} I_{1,2} I_{2,2} du = O \left(\varepsilon \int_{d_*}^N \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right) e^{i\lambda \frac{\log u}{\log N}}}{u^2 \log u} du \right) = O(\varepsilon), \\
J_{1,3,2} &= - \int_{d_*}^N I_{0,1} I_{1,3} I_{2,2} du = O \left(\varepsilon \int_{d_*}^N \frac{e^{i\lambda \frac{\log u}{\log N}}}{u^3 \log u} du \right) = O(\varepsilon), \\
J_{2,1,2} &= - \int_{d_*}^N I_{0,2} I_{1,1} I_{2,2} du = O \left(\varepsilon \int_{d_*}^N \frac{e^{i\lambda \frac{\log u}{\log N}}}{u \log^2 u} du \right) = O(\varepsilon), \\
J_{2,2,2} &= - \int_{d_*}^N I_{0,2} I_{1,2} I_{2,2} du = O \left(\varepsilon \int_{d_*}^N \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right) e^{i\lambda \frac{\log u}{\log N}}}{u^2 \log^2 u} du \right) = O(\varepsilon), \\
J_{2,3,2} &= - \int_{d_*}^N I_{0,2} I_{1,3} I_{2,2} du = O \left(\varepsilon \int_{d_*}^N \frac{e^{i\lambda \frac{\log u}{\log N}}}{u^3 \log^2 u} du \right) = O(\varepsilon), \\
J_{1,1,k+2} &= - \int_{d_*}^N I_{0,1} I_{1,1} I_{2,k+2} du = O \left(\varepsilon \int_{d_*}^N \frac{du}{u^2 \log u} \right) = O(\varepsilon), \\
J_{1,2,k+2} &= - \int_{d_*}^N I_{0,1} I_{1,2} I_{2,k+2} du = O \left(\varepsilon \int_{d_*}^N \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right)}{u^3 \log u} du \right) = O(\varepsilon), \\
J_{1,3,k+2} &= - \int_{d_*}^N I_{0,1} I_{1,3} I_{2,k+2} du = O \left(\varepsilon \int_{d_*}^N \frac{du}{u^4 \log u} \right) = O(\varepsilon), \\
J_{2,1,k+2} &= - \int_{d_*}^N I_{0,2} I_{1,1} I_{2,k+2} du = O \left(\varepsilon \int_{d_*}^N \frac{du}{u^2 \log^2 u} \right) = O(\varepsilon), \\
J_{2,2,k+2} &= - \int_{d_*}^N I_{0,2} I_{1,2} I_{2,k+2} du = O \left(\varepsilon \int_{d_*}^N \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right)}{u^3 \log^2 u} du \right) = O(\varepsilon), \\
J_{2,3,k+2} &= - \int_{d_*}^N I_{0,2} I_{1,3} I_{2,k+2} du = O \left(\varepsilon \int_{d_*}^N \frac{du}{u^4 \log^2 u} \right) = O(\varepsilon).
\end{aligned}$$

A.2 $O(\varepsilon \log \varepsilon)$ terms

The analysis of these error terms yields estimates of the form $O\left(\frac{1}{\log N}\right) + O(\varepsilon \log \varepsilon)$. We shall need the special function

$$\text{Ei}(z) = - \int_{-z}^{\infty} \frac{e^{-t}}{t} dt, \quad z \in \mathbb{R},$$

where the integral is in the sense principal value due to the singularity at zero. For complex arguments $\text{Ei}(z)$ is defined by analytic continuation. Notice that $v \mapsto \text{Ei}(iv)$ is the antiderivative of $v \mapsto e^{iv}/v$. Moreover, for $\tau, w \in \mathbb{R}$,

$$\text{Ei}(\tau) = \gamma + \log \tau + O(\tau) \quad \text{as } \tau \rightarrow 0+; \quad (45)$$

$$\text{Ei}(i\tau) = \gamma + \frac{i\pi}{2} + \log \tau + O(\tau) \quad \text{as } \tau \rightarrow 0+; \quad (46)$$

$$\text{Ei}(w) = \frac{e^w}{w} \left(1 + O\left(\frac{1}{w}\right) \right) \quad \text{as } w \rightarrow \infty. \quad (47)$$

$$\text{Ei}(iw) = i\pi - \frac{e^{iw}}{w} \left(1 + O\left(\frac{1}{w}\right) \right) \quad \text{as } w \rightarrow \infty. \quad (48)$$

By (46, 48) we have

$$\begin{aligned} J_{1,1,2} &= - \int_{d_*}^N I_{0,1} I_{1,1} I_{2,2} du = -i\alpha\varepsilon \int_{d_*/\log N}^1 \frac{e^{i\lambda v}}{v} dv = -i\alpha\varepsilon \text{Ei}(i\lambda v)|_{v=d_*/\log N}^{v=1} = \\ &= -i\alpha\varepsilon (\text{Ei}(i\lambda) - \text{Ei}(i\varepsilon d_*)) = \\ &= O\left(\varepsilon \left(O(1) - \gamma - \frac{i\pi}{2} - \log \varepsilon + \log d_* + O(\varepsilon)\right)\right) = O(\varepsilon \log \varepsilon). \end{aligned}$$

Let Ci and Si be the special functions introduced in Section 6. They satisfy the estimates

$$i\text{Ci}(y) - \text{Si}(y) = O(1) \quad \text{for } |y| \geq c > 0 \quad (49)$$

$$i\text{Ci}(\tau) - \text{Si}(\tau) = i(\gamma + \log \tau) - \tau + O(\tau^2) \quad \text{as } \tau \rightarrow 0 \quad (50)$$

We have

$$J_{2,1,1} = - \int_{d_*}^N I_{0,2} I_{1,1} I_{2,1} du = O\left(\int_{d_*}^N \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1\right)}{u \log^2 u} du\right) = O\left(\tilde{J}_{2,1,1}(x)|_{\log d_*}^{\log N}\right),$$

where

$$\tilde{J}_{2,1,1}(x) = \frac{1}{x} - \frac{e^{i\varepsilon x}}{x} + \varepsilon (i\text{Ci}(\varepsilon x) - \text{Si}(\varepsilon x)).$$

By (49, 50) we get

$$J_{2,1,1} = O\left(\frac{1}{\log N}\right) + O(\varepsilon \log \varepsilon).$$

A.3 Mixed terms

Here we present the estimates for error terms of order $O(\varepsilon) + O(\frac{1}{\log N})$. In this section we assume that $3 \leq j \leq k+1$. The following integrals can be estimated using the properties (45, 47) of the exponential integral function Ei.

$$\begin{aligned}
J_{2,1,j} &= - \int_{d_*}^N I_{0,2} I_{1,1} I_{2,j} du = O\left(\int_{d_*}^N \frac{\left(e^{i\lambda(j-2)\frac{\log u}{\log N}} - 1\right)}{u^{j-1} \log^2 u} du\right) = O\left(\tilde{J}_{2,1,j}(x)\Big|_{\log d_*}^{\log N}\right), \\
J_{2,2,j} &= - \int_{d_*}^N I_{0,2} I_{1,2} I_{2,j} du = O\left(\int_{d_*}^N \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right) \left(e^{i\lambda(j-2)\frac{\log u}{\log N}} - 1\right)}{u^j \log^2 u} du\right) = O\left(\tilde{J}_{2,2,j}(x)\Big|_{\log d_*}^{\log N}\right), \\
J_{2,3,j} &= - \int_{d_*}^N I_{0,2} I_{1,3} I_{2,j} du = O\left(\int_{d_*}^N \frac{\left(e^{i\lambda(j-2)\frac{\log u}{\log N}} - 1\right)}{u^{j+1} \log^2 u} du\right) = O\left(\tilde{J}_{2,3,j}(x)\Big|_{\log d_*}^{\log N}\right),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{J}_{2,1,j}(x) &= \frac{1}{x} \left(e^{-(j-2)x} - e^{-(j-2)(1-i\varepsilon)x} \right) + \\
&\quad + (j-2) \text{Ei}(-(j-2)x) - (j-2)(1-i\varepsilon) \text{Ei}(-(j-2)(1-i\varepsilon)x), \\
\tilde{J}_{2,2,j}(x) &= \frac{1}{x} \left(e^{-(j-1-i\varepsilon)x} - e^{-(j-1)(1-i\varepsilon)x} + e^{-(j-1-(j-2)i\varepsilon)x} - e^{-(j-1)x} \right) + \\
&\quad + (j-1-i\varepsilon) \text{Ei}(-(j-1-i\varepsilon)x) - \\
&\quad - (j-1)(1-i\varepsilon) \text{Ei}(-(j-1)(1-i\varepsilon)x) + \\
&\quad + (j-1-(j-2)i\varepsilon) \text{Ei}(-(j-1-(j-2)i\varepsilon)x) - \\
&\quad - (j-1) \text{Ei}(-(j-1)x), \\
\tilde{J}_{2,3,j}(x) &= \frac{1}{x} \left(e^{-jx} - e^{-(j-(j-2)i\varepsilon)x} \right) + \\
&\quad + j \text{Ei}(-jx) - (j-(j-2)i\varepsilon) \text{Ei}(-(j-(j-2)i\varepsilon)x).
\end{aligned}$$

By (45) and (47) we get

$$\begin{aligned}
J_{2,1,j} &= O\left(\frac{1}{N^{j-2} \log N}\right) + O(\varepsilon), \\
J_{2,2,j} &= O\left(\frac{1}{N^{j-1} \log N}\right) + O(\varepsilon^2), \\
J_{2,3,j} &= O\left(\frac{1}{N^j \log N}\right) + O(\varepsilon).
\end{aligned}$$

For the remaining error terms we shall need the incomplete gamma function

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt, \quad z \in \mathbb{R}, \tag{51}$$

and defined for complex z by analytic continuation. We will only need the cases $a = 0$ and $a = -1$. It satisfies, for $w, \tau \in \mathbb{R}$ and $c, z \in \mathbb{C}$,

$$\Gamma(0, w) = e^{-w} \left(\frac{1}{w} + O\left(\frac{1}{w^2}\right) \right), \quad w \rightarrow \infty, \quad (52)$$

$$\Gamma(a, c + z\tau) = \Gamma(a, c) - \frac{e^{-c}}{c^{1-a}} z\tau + O(\tau^2), \quad \tau \rightarrow 0. \quad (53)$$

We have

$$\begin{aligned} J_{1,1,j} &= - \int_{d_*}^N I_{0,1} I_{1,1} I_{2,j} du = O\left(\int_{d_*}^N \frac{\left(e^{i\lambda(j-2)\frac{\log u}{\log N}} - 1\right)}{u^{j-1} \log u} du\right) = O\left(\left.\tilde{J}_{1,1,j}(x)\right|_{\log d_*}^{\log N}\right), \\ J_{1,2,j} &= - \int_{d_*}^N I_{0,1} I_{1,2} I_{2,j} du = O\left(\int_{d_*}^N \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right) \left(e^{i\lambda(j-2)\frac{\log u}{\log N}} - 1\right)}{u^j \log u} du\right) = O\left(\left.\tilde{J}_{1,2,j}(x)\right|_{\log d_*}^{\log N}\right), \\ J_{1,3,j} &= - \int_{d_*}^N I_{0,1} I_{1,3} I_{2,j} du = O\left(\int_{d_*}^N \frac{\left(e^{i\lambda(j-2)\frac{\log u}{\log N}} - 1\right)}{u^{j+1} \log u} du\right) = O\left(\left.\tilde{J}_{1,3,j}(x)\right|_{\log d_*}^{\log N}\right), \\ J_{1,2,1} &= - \int_{d_*}^N I_{0,1} I_{1,2} I_{2,1} du = O\left(\int_{d_*}^N \frac{\left(e^{i\lambda\frac{\log u}{\log N}} - 1\right)^2}{u^2 \log u} du\right) = O\left(\left.\tilde{J}_{1,2,1}(x)\right|_{\log d_*}^{\log N}\right), \end{aligned}$$

where

$$\begin{aligned} \tilde{J}_{1,1,j}(x) &= \Gamma(0, (j-2)x) - \Gamma(0, (j-2)(1-i\varepsilon)x), \\ \tilde{J}_{1,2,j}(x) &= -\Gamma(0, (j-1)x) - \Gamma(0, (j-1)(1-i\varepsilon)x) + \\ &\quad + \Gamma(0, (j-1-i\varepsilon)x) + \Gamma(0, (j-1-(j-2)i\varepsilon)x), \\ \tilde{J}_{1,3,j}(x) &= \Gamma(0, jx) - \Gamma(0, (j-(j-2)i\varepsilon)x), \\ \tilde{J}_{1,2,1}(x) &= -\Gamma(0, x) + 2\Gamma(0, (1-i\varepsilon)x) - \Gamma(0, (1-2i\varepsilon)x). \end{aligned}$$

From (52) and (53) we get

$$\begin{aligned} J_{1,1,j} &= O\left(\frac{1}{N^{j-2} \log N}\right) + O(\varepsilon), \\ J_{1,2,j} &= O\left(\frac{1}{N^{j-1} \log N}\right) + O(\varepsilon). \\ J_{1,3,j} &= O\left(\frac{1}{N^j \log N}\right) + O(\varepsilon). \\ J_{1,2,1} &= O\left(\frac{1}{N \log N}\right) + O(\varepsilon^2). \end{aligned}$$

The following error term estimates involve a combination of (45,47) and (52,53).

$$\begin{aligned}
J_{1,3,1} &= - \int_{d_*}^N I_{0,1} I_{1,3} I_{2,1} du = O \left(\int_{d_*}^N \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right)}{u^3 \log u} du \right) = O \left(\left| \tilde{J}_{1,3,1}(x) \right|_{\log d_*}^{\log N} \right), \\
J_{2,2,1} &= - \int_{d_*}^N I_{0,2} I_{1,2} I_{2,1} du = O \left(\int_{d_*}^N \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right)^2}{u^2 \log^2 u} du \right) = O \left(\left| \tilde{J}_{2,2,1}(x) \right|_{\log d_*}^{\log N} \right), \\
J_{2,3,1} &= - \int_{d_*}^N I_{0,2} I_{1,3} I_{2,1} du = O \left(\int_{d_*}^N \frac{\left(e^{i\lambda \frac{\log u}{\log N}} - 1 \right)}{u^3 \log^2 u} du \right) = O \left(\left| \tilde{J}_{2,2,1}(x) \right|_{\log d_*}^{\log N} \right),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{J}_{1,3,1} &= -\text{Ei}(-2x) - \Gamma(0, (2 - i\varepsilon)x), \\
\tilde{J}_{2,2,1} &= -\text{Ei}(-x) + 2(1 - i\varepsilon)\Gamma(-1, (1 - i\varepsilon)x) - (1 - 2i\varepsilon)\Gamma(-1, (1 - 2i\varepsilon)x) - \frac{1}{xe^x}, \\
\tilde{J}_{2,3,1} &= 2\text{Ei}(-2x) + \frac{1}{xe^{2x}} - (2 - i\varepsilon)\Gamma(-1, (2 - i\varepsilon)x).
\end{aligned}$$

We get

$$\begin{aligned}
J_{1,3,1} &= O \left(\frac{1}{N \log N} \right) + O(\varepsilon), \\
J_{1,2,2} &= O \left(\frac{1}{N \log^2 N} \right) + O(\varepsilon^2), \\
J_{2,3,1} &= O \left(\frac{1}{N} \right) + O(\varepsilon).
\end{aligned}$$

Combining all the error terms discussed in this appendix we obtain the desired estimate (38).

References

- [1] K. Alladi. An Erdős-Kac theorem for integers without large prime factors. *Acta Arith.*, 49(1):81–105, 1987.
- [2] M. Avdeeva, D. Li, and Ya.G. Sinai. New limiting distributions for the Möbius function. *preprint*.
- [3] R. C. Baker. The square-free divisor problem. II. *Quart. J. Math. Oxford Ser. (2)*, 47(186):133–146, 1996.
- [4] P.T. Bateman. Proof of a conjecture of Grosswald. *Duke Math. J.*, 25:67–72, 1957.

[5] F. Cellarosi and Ya. G. Sinai. The Möbius function and statistical mechanics. *Bull. Math. Sci.*, 1(2):245–275, 2011.

[6] F. Cellarosi and Ya.G Sinai. Non-standard limit theorems in Number Theory. *to appear in ‘Prokhorov Festschrift’*.

[7] N. G. de Bruijn. The asymptotic behaviour of a function occurring in the theory of primes. *J. Indian Math. Soc. (N.S.)*, 15:25–32, 1951.

[8] N. G. de Bruijn. On the number of positive integers $\leq x$ and free of prime factors $> y$. *Nederl. Acad. Wetensch. Proc. Ser. A.*, 54:50–60, 1951.

[9] N. G. de Bruijn. On the number of positive integers $\leq x$ and free prime factors $> y$. II. *Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math.*, 28:239–247, 1966.

[10] J.-M. De Koninck and D. Hensley. Sums taken over $n \leq x$ with prime factors $\leq y$ of $z^{\Omega(n)}$, and their derivatives with respect to z . *J. Indian Math. Soc. (N.S.)*, 42(1-4):353–365 (1979), 1978.

[11] H. Delange. Sur des formules de Atle Selberg. *Acta Arith.*, 19:105–146. (errata insert), 1971.

[12] K. Dickman. On the frequency of numbers containing primes of a certain relative magnitude. *Ark. Mat. Astr. Fys.* 22A:1–14, 1930.

[13] P. Erdős. *Wiskundige Opgaven met de Oplossingen*, 21:Problem and Solution Nr. 136, 1963.

[14] P. Erdős and M. Kac. The Gaussian law of errors in the theory of additive number theoretic functions. *Amer. J. Math.*, 62:738–742, 1940.

[15] A. Granville. Smooth numbers: computational number theory and beyond. In *Algorithmic number theory: lattices, number fields, curves and cryptography*, volume 44 of *Math. Sci. Res. Inst. Publ.*, pages 267–323. Cambridge Univ. Press, Cambridge, 2008.

[16] E. Grosswald. The average order of an arithmetic function. *Duke Math. J.*, 23:41–44, 1956.

[17] G. Hanrot, G. Tenenbaum, and J. Wu. Moyennes de certaines fonctions multiplicatives sur les entiers friables. II. *Proc. Lond. Math. Soc. (3)*, 96(1):107–135, 2008.

[18] G. H. Hardy and S. Ramanujan. The normal number of prime factors of a number n [Quart. J. Math. **48** (1917), 76–92]. In *Collected papers of Srinivasa Ramanujan*, pages 262–275. AMS Chelsea Publ., Providence, RI, 2000.

[19] D. Hensley. The sum of $\alpha^{\Omega(n)}$ over integers $n \leq x$ with all prime factors between α and y . *J. Number Theory*, 18(2):206–212, 1984.

[20] D. Hensley. The distribution of $\Omega(n)$ among numbers with no large prime factors. In *Analytic number theory and Diophantine problems (Stillwater, OK, 1984)*, volume 70 of *Progr. Math.*, pages 247–281. Birkhäuser Boston, Boston, MA, 1987.

- [21] A. Hildebrand. Integers free of large prime factors and the Riemann hypothesis. *Mathematika*, 31(2):258–271 (1985), 1984.
- [22] A. Hildebrand. On the number of prime factors of integers without large prime divisors. *J. Number Theory*, 25(1):81–106, 1987.
- [23] A. Hildebrand. Quantitative mean value theorems for nonnegative multiplicative functions. II. *Acta Arith.*, 48(3):209–260, 1987.
- [24] M. N. Huxley. A note on short exponential sums. In *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989)*, pages 217–229, Salerno, 1992. Univ. Salerno.
- [25] E. Landau. Über die Verteilung der Zahlen welche aus v Primfaktoren zusammengesetzt sind. *Nach. Ges. Wiss. Göttingen*, pages 361–382, 1911.
- [26] F. Mertens. Über einige asymptotische Gesetze der Zahlentheorie.
- [27] F. Mertens. Ein Beitrag zur analytischen Zahlentheorie. Über die Vertheilung der Primzahlen. *J. reine angew. Math.*, 78:46–62, 1874.
- [28] A. Selberg. Note on a paper by L. G. Sathe. *J. Indian Math. Soc. (N.S.)*, 18:83–87, 1954.
- [29] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 46 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Translated from the second French edition (1995) by C. B. Thomas.
- [30] G. Tenenbaum and J. Wu. Moyennes de certaines fonctions multiplicatives sur les entiers friables. *J. Reine Angew. Math.*, 564:119–166, 2003.
- [31] G. Tenenbaum and J. Wu. Moyennes de certaines fonctions multiplicatives sur les entiers friables. III. *Compos. Math.*, 144(2):339–376, 2008.
- [32] G. Tenenbaum and J. Wu. Moyennes de certaines fonctions multiplicatives sur les entiers friables. IV. In *Anatomy of integers*, volume 46 of *CRM Proc. Lecture Notes*, pages 129–141. Amer. Math. Soc., Providence, RI, 2008.